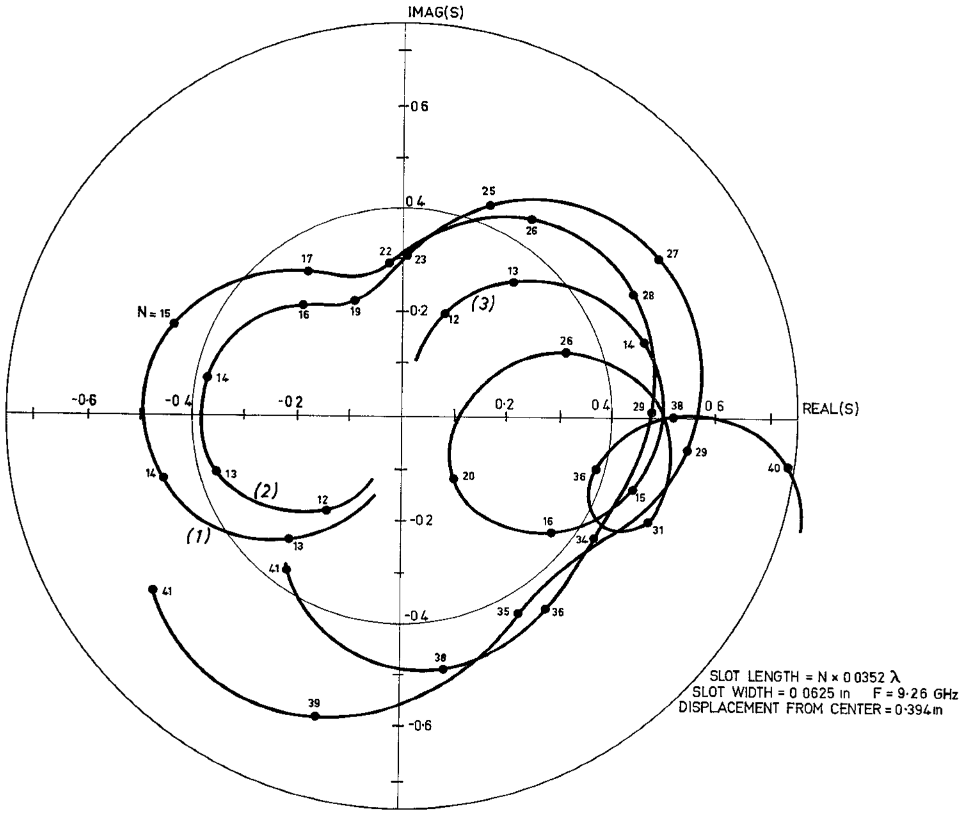

Fig. 6. Variation of coupling coefficient and reflection coefficient with slot length. (*1*), S_{11} of a shunt slot in broadwall coupling two identical waveguides; (*2*), S_{11} of a shunt slot in broadwall radiating into free space; (*3*), coupling coefficient of a shunt slot in broadwall coupling two identical waveguides.

and provides as its solution the electric field at the aperture, in addition to the various parameters of interest such as coupling coefficient, junction impedance, etc.

ACKNOWLEDGMENT

The author gratefully acknowledges discussions with G. K. Cambrell and C. T. Carson concerning this work.

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A Generalized Locking Equation for Oscillators

JOHN P. QUINE

***Abstract*—Locking equations are derived which account for nonsinusoidal device waveforms. Locking bandwidth is related to Q values and device voltage amplitudes. Effective Q values are calculated for cavities having tuned and untuned harmonics.**

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Adler [1] has derived a relation between the reflection phase shift and the gain of locked oscillators based on the assumption of a sinusoidal device voltage waveform. However, to achieve high efficiency in present-day solid-state microwave power sources, it is required in many cases that the device current and voltage waveforms contain strong harmonic components. In these cases, Adler's equation does not apply. In this short paper, the derivation of a generalized locking equation is presented which accounts for the presence of strong harmonic components, and allows the prediction of the locking bandwidth for an arbitrary cavity configuration and for arbitrary device waveforms.

A single-valued static i–v characteristic is assumed. For this case, the area described by the instantaneous operating point as it moves along the i–v curve during one fundamental period is zero [2]. Thus

$$\int_0^T i\,dv = 0, \tag{1}$$

where $T = 1/f = 2\pi/\omega$ is the fundamental period of oscillation, and

$$v = \sum_{n=1}^{\infty} V_n \sin(2n\pi ft + \alpha_n) \tag{2}$$

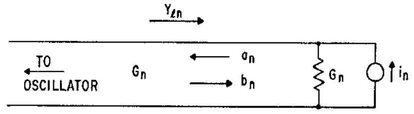
$$i = \sum_{n=1}^{\infty} I_n \sin(2n\pi ft + \beta_n) \tag{3}$$

$$dv = \sum_{n=1}^{\infty} n\omega V_n \cos(2n\pi ft + \alpha_n)dt. \tag{4}$$

Performing the integration indicated by (1) yields Groszkowski's result [2]:

$$\sum_{n=1}^{\infty} n\,|V_n|\,|I_n|\,\sin(\alpha_n - \beta_n) = 0. \tag{5}$$

Equation (5) states that the total reactive power flow is zero, and can be expressed as



$$Y_{\ell n} = G_n \frac{b_n - a_n}{b_n + a_n} = G_n \left[\frac{\Gamma_n - 1}{\Gamma_n + 1}\right]$$

$$Y_{\ell n} = G_{\ell n} + j B_{\ell n} \quad ; \Gamma_n = r_n e^{j\theta_n}$$

$$G_{\ell n} = \frac{r_n^2 - 1}{1 + 2r_n \cos\theta_n + r_n^2} G_n$$

$$B_{\ell n} = \frac{2r_n \sin\theta_n}{1 + 2r_n \cos\theta_n + r_n^2} G_n$$

Fig. 1. Derivation of admittance Y_{ln}.

$$\sum_{n=1}^{\infty} nE_n^2 B_n = 0 \tag{6}$$

where $E_n = |V_n/V_1|$ and B_n is the susceptance presented to the device by the cavity at the nth harmonic frequency nf. The dual relation

$$\sum_{n=1}^{\infty} nH_n^2 X_n = 0 \tag{7}$$

can also be derived where $|H_n| = |I_n/I_1|$ and X_n is the reactance presented to the device by the cavity.

Equation (6) or (7) defines the general resonance condition that must be satisfied when any linear or nonlinear, active or passive, device represented by a "resistive" single-valued i–v characteristic interacts with a cavity characterized by the susceptances B_n. For a linear device, the only possible solution is for each B_n to be zero. On the other hand, for a nonlinear device, the condition $B_n = 0$ is a permissible solution, but not a necessary condition in general. With harmonic voltages present, the nonlinear device, although represented by a nominally resistive i–v characteristic, can appear as a susceptance, and in this case the B_n are not required to be individually zero in general.

The origin of these susceptances can be more clearly understood by considering the device to be driven directly and simultaneously by hypothetical zero-impedance generators having voltages given by (2). If all the α_n are made equal to $\pm\pi/2$, then (2) shows that the device voltage waveform is symmetrical about $t=0$. In this case, the resulting current waveform for the nonlinear (and linear) device must also be symmetrical and I_n must be in phase (or 180° out of phase) with V_n; under these conditions, the device is "resistive" at each frequency. On the other hand, if any of the α_n are not made equal to $\pm\pi/2$, and if the device is nonlinear, the resulting nonsymmetrical voltage waveform across the device can cause I_n to have any phase with respect to V_n; under these conditions, the device can appear "susceptive" at the frequency nf.

In the case of an actual oscillator or amplifier, the cavity admittances establish the phase between V_n and I_n, the operating point and free-running frequency of the oscillator (or center frequency of the amplifier) adjusting in order to satisfy (6). If all the B_n are adjusted to be zero at a particular operating frequency, the arguments presented in the preceding paragraph show that the voltage waveform across the device must be symmetrical, e.g., square or half-sinusoidal. It was shown previously that such waveforms can lead to enhanced efficiency [3], [4]. As the operating frequency is changed by changing the frequency of an applied signal, it is clear that the B_n may no longer have zero values, and the voltage waveforms may no longer be symmetrical.

To derive a locking equation, it is noted that B_n in (6) comprises the actual "cold"-cavity susceptance B_{cn} plus the susceptance B_{ln} associated with a locking signal that may be applied at any or all of the harmonic frequencies nf [5]. Fig. 1 shows how B_{ln} can be calculated. It is assumed that at each frequency nf the device is connected to a match-terminated transmission line of characteristic admittance G_n, and that an applied locking power represented by a generator of constant current $i_n = \sqrt{8P_nG_n}$ results in a wave a_n incident on the device and a reflected wave b_n. The admittance Y_{ln} looking from the device can be determined by considering b_n to be incident on the load and a_n reflected from the load. Equation (6) can therefore be written as

$$\sum_{n=1}^{\infty} \frac{2r_nG_n \sin\theta_n}{1 + 2r_n \cos\theta_n + r_n^2} nE_n^2 + \sum_{n=1}^{\infty} nE_n^2 B_{cn} = 0 \tag{8}$$

where r_n and θ_n are the magnitude and the phase of the complex reflection gain experienced by a locking signal applied at frequency nf.

If a locking signal is applied only at the fundamental frequency f, and if this signal is small ($r_1 \gg 1$), then (8) reduces to

$$\frac{2G_1 \sin\theta_1}{r_1} \doteq -B_{1c} - \sum_{n=2}^{\infty} nE_n^2 B_{cn}. \tag{9}$$

By equating B_{c1} to the susceptance of a single-tuned LC cavity, and by neglecting the harmonic voltages, (9) reduces to a form equivalent to that of Adler. However, the two results differ by a factor of two, as was explained previously [5], [6].

It can be shown that (8) also applies for stable reflection amplifiers. Thus the reflection phase shift θ_n can be calculated for a specified reflection gain r_n for a stable amplifier or for a locked oscillator.

The above results can be applied to specific cavity configurations.

Case I (Untuned Harmonics)

In this case, the cavity comprises a frequency-independent lumped inductance L and capacitance C in parallel. The cold-cavity susceptance B_{cn} is given by

$$B_{cn} = 2\pi f_0 C \left[\frac{nf}{f_0} - \frac{f_0}{nf}\right] \tag{10}$$

where $f_0 = 1/(2\pi\sqrt{LC})$. After employing (10) into (9), one can obtain the free-running frequency f_f by setting r_1 equal to infinity. The result is

$$\frac{f_f}{f_0} - \frac{f_0}{f_f} = -\sum_{n=2}^{\infty} nE_n^2 \left[\frac{nf_f}{f_0} - \frac{f_0}{nf_f}\right]$$

which can be manipulated to obtain the form

$$\left(\frac{f_0}{f_f}\right)^2 = \frac{1 + \sum_{n=2}^{\infty} n^2E_n^2}{1 + \sum_{n=2}^{\infty} E_n^2}. \tag{11}$$

Equation (11) was also derived by van der Pol [7], [8] from the differential equation for the LC oscillator. Thus for the special case of a simple LC oscillator, it has been shown here that Groszkowski's more general approach leads to the same result obtained by van der Pol. Equation (11) shows that f_f can be substantially less than f_0 depending on the harmonic content, as was discussed again recently [8], [9].

Fig. 2 shows values of $\sin\theta_1$ calculated from (9) using the value of B_{cn} given by (10), and assuming a half-sinusoidal voltage waveform as an approximation for a limited space-charge accumulation (LSA) relaxation-mode oscillator [8]. In this case, only even harmonics occur, and the harmonic voltages relative to that of the fundamental are 0.424, 0.0849, 0.0364, 0.0202, and 0.0129 for $n=2$, 4, 6, 8, and 10, respectively. The parameter N in Fig. 2 indicates the highest harmonic included. The locking bandwidth corresponds to $|\sin\theta_1| < 1$, and the data show that the presence of the harmonics can cause a substantial reduction in bandwidth. The data also show that the bandwidth reduction obtained when several harmonics are present is not appreciably greater than when only the second harmonic is present. The shifts in center frequencies correspond to the values calculated from (11).

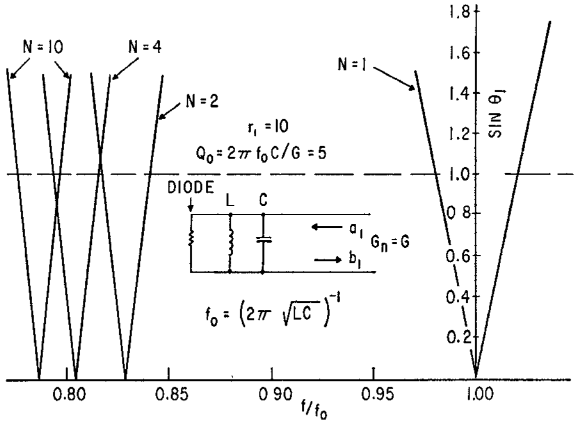


Fig. 2. Bandwidth and frequency shift of LC circuit.

Case II (Tuned Harmonics)

A hypothetical cavity can be considered comprising a frequency-independent capacitance C tuned at each harmonic by an inductance L_n which is somehow made different for each harmonic. For this model, if f is the fundamental frequency, then

$$B_{cn} = 2\pi f_n C\left[\frac{nf}{f_n} - \frac{f_n}{nf}\right] \tag{12}$$

where f_n is the resonant frequency for the nth harmonic. If $f_n = nf_1$, then (12) shows that $B_{cn} = nB_{c1} = 2n\pi f_1 C(f/f_1 - f_1/f) = nQ_1G_1(f/f_1 - f_1/f)$. Under these conditions, B_{cn} is zero at midband for all harmonics.

For this hypothetical cavity, therefore, (9) becomes

$$\sin\theta_1 = -\frac{Q_1 r_1}{2}\left[1 + \sum_{n=2}^{\infty} n^2E_n^2\right]\left(\frac{f}{f_1} - \frac{f_1}{f}\right). \tag{13}$$

It is noted that no detuning occurs, i.e., $\sin\theta_1 = 0$ at $f = f_1$. From the form of (13), it is seen that the effective Q value is given by

$$Q_{\text{eff}} = Q_1\left[1 + \sum_{n=2}^{\infty} n^2E_n^2\right]. \tag{14}$$

Assuming only the fundamental and the second harmonic of a half-sine-wave spectrum are present, the value of Q_{eff} is $1.72Q_1$. If the fourth harmonic is present as well as the second, then Q_{eff} is $1.83Q_1$.

The results of this short paper, although based on an idealized model, have been found useful in explaining the narrow-locking bandwidths obtained with LSA relaxation-mode oscillators which have approximately half-sine-wave voltage waveforms [10]. The narrow bandwidth obtained with TRAPATT oscillators having approximately square current waveforms can also be understood on the same basis.

Acknowledgment

The author wishes to thank Dr. J. R. Eshbach and Dr. S. P. Yu for their helpful review comments.

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Comment on "Varactor Q Measurement"

EUGENE W. SARD and JAMES M. ROE

In the above letter,[1] and in a previous one by Houlding [1] it has been stated that less than six parameters are sufficient to characterize a lossy two-port network coupling into a varactor junction. It is my contention, however, that erroneous conclusions can thereby be drawn as to the varactor's Q. Specifically, Roe's method will be applied to a theoretical varactor and shown to give ambiguous results, which are attributed to Roe's use of less than six parameters to describe the lossy coupling network.

The theoretical varactor chosen is the one considered previously in [2], with $n = \frac{1}{2}$, $\phi = 1.2$ V, total $R_S = 1\ \Omega$, and $Q_0' = 10$ (the varactor Q at the matched bias point $v = 0$ V). Fig. 1 shows three two-port networks coupling into the varactor junction to give identical admittance circles with varying bias, corresponding to the values of $\alpha = 3$, 0, and -3. There are an infinite number of such networks corresponding to different values of α, the normalized series load reactance when biased for minimum standing-wave ratio (SWR). Fig. 1(b) ($\alpha = 0$) is the same as [2, fig. 10(b)], and is the simplified configuration considered by Roe. Fig. 1(a) and (c) are new and are derived from [2, eqs. (35)–(38)]. The admittance circles are centered on the real axis, intersecting it at $g = 1/9$ and 1.

Table I summarizes the calculation of the values of Roe's Q_m for the three networks, and Fig. 2 shows Q_m plotted versus $\phi - v$. Only the Q_m points corresponding to the $\alpha = 0$ network fit Roe's theoretical curve. Thus, unless one is lucky enough to have $\alpha = 0$ when measuring the varactor Q, the results using Roe's method will be erroneous. In contrast there was no theoretical difficulty in handling values of $\alpha \neq 0$ [2], which used six parameters to characterize the lossy two-port network coupling into the varactor junction.

Reply by James M. Roe[2]

That five parameters are sufficient to map the entire output impedance plane of a lossy two-port network into the input impedance plane is obvious by inspection. The network of Fig. 1[1] will map the R = constant line of the output plane into a circle in the input plane which can be described by the equation

$$(R_{\text{in}} - h)^2 + (X_{\text{in}} - k)^2 = r^2. \tag{1}$$

In effect, Sard claims that the same circle can be obtained by adding an arbitrary reactance term X in series to the circle described by

$$(R_{\text{in}} - h)^2 + (X_{\text{in}} - k')^2 = r^2 \tag{1'}$$

where

$$k' = k - X.$$

It should be clear that one can extend this process to the inclusion of an arbitrary series resistance term, also. In fact, by alternating between the input impedance and admittance planes, one can add arbitrary elements forever, and still end up with the same circle. I do not see that six elements represents a special class of transformations.

The ambiguity demonstrated by Sard does not result from the usage of the simplified transformation, but rather from Sard's usage of the *same* varactor law with the arbitrary series reactance term. Each of the three examples shown above gives data points which lie on the same circle, but the range of the arcs covered is different. Of course, only one arc is observed, and the problem is to establish which circuit interpretation is correct. Sards' method [2] begins with the premise that the varactor law is known (he assumes that the varactor model determined at a low frequency applies at frequencies orders of magnitude greater), and determines the appropriate transformation. My method[1] uses the simple transformation, and determines the

Manuscript received November 15, 1971.
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[1] J. M. Roe, *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-19, pp. 728–729, Aug. 1971.
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